Unit IV Nonlinear Waves, Shocks and Turbulence – An Introduction

Previously discussed:

→ basic waves in MHD, i.e. structure of MHD 'stiffness matrix'.

→ δW and MHD instabilities (an introduction)

Now, are concerned with evolving waves and instabilities i.e. what happens? →

- nonlinear amplification of MHD waves, wavebreaking
- shocks and collisionless shocks in MHD,
- turbulence

Flow of development is:

waves (perturbation) → [instabilities, nonlinear waves] (amplification) → [breaking, shocks, collisionless shocks, turbulence] (nonlinear evolution)

→ [heating, mixing, transport, structure formation, disruption, ⋮] ↔ (practical consequences)

Proceed via:

i.) Nonlinear Waves
  a.) Wave Action and Eikonal Theory
  b.) Wave Amplification and Breaking
  c.) Disparate Scale Interaction

ii.) Shocks and collisionless shocks
  a.) shocks in kinematic waves
  b.) shocks in fluids and MHD
  c.) collisionless shocks in plasmas

iii.) Fluid and MHD Turbulence - An Introduction

then, ... ⇒ Applications to Laboratory and Solar Plasmas

① → current and magnetic configurations (δW with J_0)

② → ∇p stability of confinement devices

③ → magnetic fields and buoyancy in the sun.

Read: ① Kulsrud 5.5, 5.6
② Whitham, Chapt. 11
③ Landau, Lifshitz Fluids Chapt.

→ Nonlinear Waves

→ have considered plane waves in uniform media

i.e. $\mathcal{E} \sim \mathcal{E}_{k}\, e^{i(\underline{k}\cdot\underline{x}-\omega t)}$

→ what if media non-uniform, but slowly varying

i.e. $$\frac{1}{c^2}\frac{\partial^2}{\partial t^2}\tilde{\rho} = \nabla^2\tilde{\rho}$$ (acoustics)

with $c^2 = c_0^2/n^2(\underline{x})$

↳ index of refraction (can be time dependent)

then for $\left|\frac{\nabla n}{n}\right| < |\underline{k}|$, can write

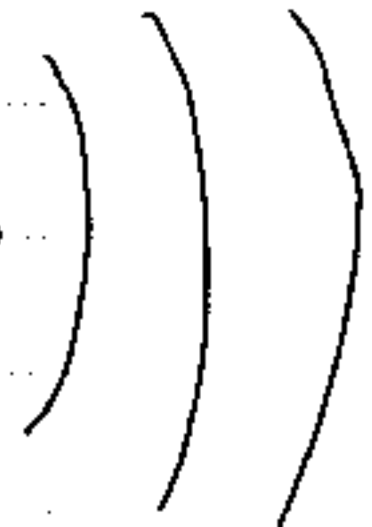
$$\tilde{\rho} = \rho_0\, e^{i\phi(\underline{x},t)}$$

where $\phi \sim O(1/\epsilon)$
→ phase contains fastest variation

then have:

$$\boxed{\frac{n(x)^2}{c_0^2}\left(\frac{\partial\phi}{\partial t}\right)^2 = \left(\nabla\phi\right)^2}$$

→ eikonal equation for phase front function ϕ

i.e. ⇒  iso-ϕ surfaces ⇒ $\nabla\phi$ ⇒ direction of propagation clear analogy with plane waves $\Rightarrow$

$$\underline{\nabla}\phi \leftrightarrow \underline{k}$$

$$-\frac{\partial \phi}{\partial t} \leftrightarrow \omega$$

[if n time independent, $\omega = \text{const.}$ for linear wave]

so eikonal equation is : $\boxed{\frac{n(x)^2}{c_0^2}\omega^2 = k^2}$

↳ local dispersion relation

so, have for medium with no explicit time dependence,

$$d\phi = \frac{\partial \phi}{\partial \underline{x}} \cdot d\underline{x} + \frac{\partial \phi}{\partial t} dt$$

$$= \underline{k}(x) \cdot d\underline{x} - \omega(\underline{k}, \underline{x})\, dt$$

↳ via Eikonal Equation

$$\therefore \frac{d\phi}{dt} = \underline{k}(\underline{x}) \cdot \frac{d\underline{x}}{dt} - \omega(\underline{k}, \underline{x})$$

$\Rightarrow$

$$\Phi = \int dt \left[\underline{k}(x) \cdot \dot{\underline{x}} - \omega \right]$$

but recall:

$$\underbrace{\mathcal{S}}_{\text{action}} = \int dt \left(p\dot{q} - \underbrace{H}_{\text{Hamiltonian}} \right) \qquad \text{and} \qquad \delta S = 0 \Rightarrow \text{equations of motion}$$

Can immediately note analogy:

Hamiltonian Dynamics	Rays/Eikonal Theory
$\underline{p}$ → momentum ($= \partial L/\partial \dot{\underline{q}}$)	$\underline{k}$ ($= \underline{\nabla}\phi$)
$\underline{q}$ → gen. coord	$\underline{x}$ (phase front position)
H → Hamiltonian	ω (frequency)
ϕ → phase function	$\mathcal{S}$ → action

and recall Hamilton-Jacobi Equation:

$$\frac{\partial S}{\partial t} + H\left(q, \frac{\partial S}{\partial q}\right) = 0$$

⇒ phase evolution equation:

$$\frac{\partial \phi}{\partial t} + \omega(\underline{k}, \underline{x}) = 0$$

$$\underline{k} = \underline{\nabla}\phi$$

exact isomorphism

$\therefore$ just as advance Hamiltonian variables in time via Hamilton's Eqn. of Motion, i.e.

$$\frac{dp}{dt} = -\frac{\partial H}{\partial \underline{q}}, \quad \frac{dq}{dt} = \frac{\partial H}{\partial p}$$

then can advance $\underline{k}$ and $\underline{x}$ analogously by:

Ray, Eikonal Equations

$$\boxed{\frac{d\underline{k}}{dt} = -\frac{\partial \omega}{\partial \underline{x}}; \quad \frac{d\underline{x}}{dt} = \frac{\partial \omega}{\partial \underline{k}} = \underline{V}_{gr}}$$

Snell's Law — group velocity

$\underline{k} = \underline{\nabla}\phi \rightarrow$ phase front orientation

$\underline{x} \rightarrow$ position of phase front.

check: If analogy is valid, should be able to derive eikonal equations from $\delta \underline{\Phi} = 0$

$$\underline{\Phi} = \int dt \left[\underline{k} \cdot \dot{\underline{x}} - \omega(\underline{k}, \underline{x}) \right]$$

$$\delta \Phi = \int dt \left[\underline{k} \cdot \delta \dot{\underline{x}} + \delta \underline{k} \cdot \dot{\underline{x}} - \left(\frac{\partial \omega}{\partial \underline{x}} \cdot \delta \underline{x} + \frac{\partial \omega}{\partial \underline{k}} \cdot \delta \underline{k} \right) \right]$$

$\delta \underline{x} = \delta \underline{k} = 0$ at end-points

$\Rightarrow$

$$\delta \Phi = \int dt \left[\left(\underline{k} \cdot \frac{d}{dt} \delta \underline{x} - \frac{\partial \omega}{\partial \underline{x}} \cdot \delta \underline{x} \right) + \left(\dot{\underline{x}} - \frac{\partial \omega}{\partial \underline{k}} \right) \cdot \delta \underline{k} \right]$$

ibp

$$\delta \Phi = \cancel{\underline{k} \cdot \delta \underline{x}} \Big|_{t_1}^{t_2} + \int dt \left[\left(\frac{d\underline{k}}{dt} + \frac{\partial \omega}{\partial \underline{x}} \right) \cdot \delta \underline{x} + \left(\dot{\underline{x}} - \frac{\partial \omega}{\partial \underline{k}} \right) \cdot \delta \underline{k} \right]$$

$\Rightarrow$ $\delta \underline{x}, \delta \underline{k} \neq 0$, $\Rightarrow$

$$\frac{d\underline{k}}{dt} = -\frac{\partial \omega}{\partial \underline{x}}, \qquad \frac{d\underline{x}}{dt} = \frac{\partial \omega}{\partial \underline{k}}$$

$\underline{\underline{so}}$ → eikonal equations are Hamiltonian equations

→ eikonal equations extremize Φ.

→ its eikonal equations satisfy Liouville's Theorem

$\underline{\underline{i.e.}}$ "flow" in phase space $\underline{k}$, $\underline{x}$ is incompressible

$$\frac{\partial}{\partial \underline{k}} \cdot \frac{d\underline{k}}{dt} + \frac{\partial}{\partial \underline{x}} \cdot \frac{d\underline{x}}{dt} = -\frac{\partial^2 \omega}{\partial \underline{k}\,\partial \underline{x}} + \frac{\partial^2 \omega}{\partial \underline{x}\,\partial \underline{k}}$$

$$= 0$$

∴ so if define wave density $\rho(\underline{k}, \underline{x}, t)$

then $$\frac{\partial \rho}{\partial t} + \nabla_\Gamma \cdot (\underline{V}_\Gamma \rho) = 0$$

but $\underline{\nabla} \cdot \underline{V}_\Gamma = 0$ $\qquad \underline{V}_\Gamma = \left[\frac{d\underline{x}}{dt}, \frac{d\underline{k}}{dt}\right]$

$$\Rightarrow \quad \frac{\partial \rho}{\partial t} + V_\Gamma \cdot \underline{\nabla}_\Gamma \rho = 0$$

$$\frac{\partial \rho}{\partial t} + \underline{V}_{gr} \cdot \frac{\partial}{\partial \underline{x}} \rho - \frac{\partial \omega}{\partial \underline{x}} \cdot \frac{\partial \rho}{\partial \underline{k}} = 0$$

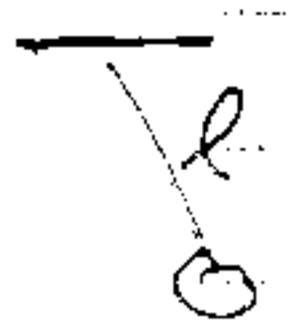
$\Rightarrow$ Vlasov-like equation for evolution of ρ!

but... what is ρ?

$\rightarrow$ physical argument:

have $\dfrac{d\rho}{dt} = 0$ $\Rightarrow$ conservation/invariance principle

Now, recall for oscillator with slowly varying parameters



$$\ell = \ell(t)$$

$$\frac{1}{\ell(t)} \frac{d\ell}{dt} < \omega = \sqrt{g/\ell}$$

then $\dfrac{d}{dt}\left(E/\omega\right) = 0$

$E/\omega \equiv$ Action (dims energy * time)

$$E = 2 \cdot \tfrac{1}{2} m \omega^2 \ell^2 \Theta^2 = m g \ell \Theta^2$$

so $$\frac{E}{\omega} = m\sqrt{g}\, \ell^{3/2}\, \theta^2$$

$$\Rightarrow d(E/\omega) = 0 \Rightarrow \frac{3}{2}\ell^{1/2}\frac{d\ell}{dt} + \ell^{3/2}\frac{d\theta^2}{dt}$$

$$d\theta^2/dt = -3/2\,\frac{1}{\ell}\frac{d\ell}{dt}$$

→ ℓ shortened ($\dot{\ell}<0$), amplitude increased

→ ℓ lengthened, amplitude decreased.

Now, for waves argue analogue of action is wave action density $\mathcal{E}/\omega = N$

$\mathcal{E}$ = energy density
N = action density

so wave kinetic equation is:

$$\boxed{\frac{\partial N}{\partial t} + \underline{v}_{gr}\cdot\frac{\partial N}{\partial \underline{x}} - \frac{\partial \omega}{\partial \underline{x}}\cdot\frac{\partial N}{\partial \underline{k}} = 0}$$

and analogy with Vlasov equation is evident, i.e.

$$\frac{\partial f}{\partial t} + v\frac{\partial f}{\partial x} + \frac{q}{m}E\frac{\partial f}{\partial v} = 0.$$

$$\frac{d\underline{k}}{dt} = -\left(\frac{\partial\omega}{\partial x}\right)_{\underline{k}} \Rightarrow \text{ no conflict with } \frac{\partial \underline{k}}{\partial t} = -\frac{\partial \omega}{\partial \underline{x}}$$

→ Now, if system independent of time, have:

$$\partial\omega/\partial t = 0$$

so

$$\frac{d\omega}{dt} = \cancel{\frac{\partial\omega}{\partial t}} + \frac{\partial\omega}{\partial \underline{k}}\cdot\frac{d\underline{k}}{dt} + \frac{\partial\omega}{\partial \underline{x}}\cdot\frac{d\underline{x}}{dt}$$

$$= -\frac{\partial^2\omega}{\partial \underline{k}\,\partial \underline{x}} + \frac{\partial^2\omega}{\partial \underline{k}\,\partial \underline{x}} = 0 \checkmark$$

$\therefore$

$$\left.\frac{dN}{dt}\right)_{\text{rays}} = 0 \Rightarrow \left.\frac{d}{dt}\left[\frac{\mathcal{E}}{\omega}\right]\right)_{\text{rays}} = 0$$

$\Rightarrow$

$$\left.\frac{1}{\omega}\frac{d\mathcal{E}}{dt}\right)_{\text{rays}} - \cancel{\left.\frac{\mathcal{E}}{\omega^2}\frac{d\omega}{dt}\right)_{\text{rays}}} = 0$$

$$\Rightarrow \quad \frac{\partial\mathcal{E}}{\partial t} + \underline{v}_{gr}\cdot\underline{\nabla}\,\mathcal{E} - \frac{\partial\omega}{\partial \underline{x}}\cdot\underline{\nabla}_k\,\mathcal{E} = 0$$

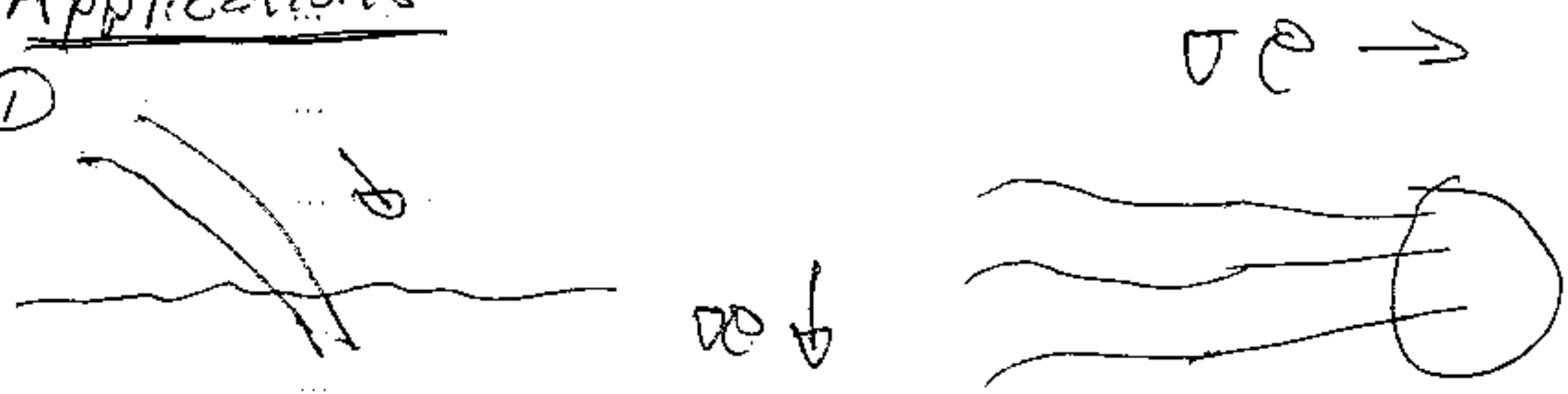
and Liouville and integrate over $\underline{k}$ $\Rightarrow$

$$\frac{d\mathcal{E}}{dt} = \frac{\partial \mathcal{E}}{\partial t} + \nabla \cdot \left[\underline{V}_{gr} \mathcal{E} \right] = 0$$

applies to conservative case.

Applications

①



Alfven wave packet incident on region with density increasing, field fixed.

i.e.

$$\nabla \cdot (V_{gr} \mathcal{E}) = 0 \qquad \underline{B} = B \hat{z}$$

$$\frac{\partial}{\partial z} \left(V_A \mathcal{E} \right) = 0 \qquad V_A = B / \sqrt{4\pi \rho(z)}$$

$$\underbrace{V_{A\infty} \mathcal{E}_\infty}_{\text{Inflow } I} = V_A(z) \, \mathcal{E}(z)$$

$$I = V_A(z)\,\mathcal{E}(z)$$

$$= V_{A\infty}\sqrt{\frac{\rho_\infty}{\rho(z)}}\;\mathcal{E}(z)$$

$$\Rightarrow \mathcal{E}(z) = \left(\rho(z)/\rho_\infty\right)^{1/2}\mathcal{E}_\infty$$

→ wave energy density increases in high density region

→ point is $V_{gr}\,\mathcal{E}$ = const

$V_{gr} = V_A \downarrow$ while $\rho \uparrow$, so $\mathcal{E}$ does increase

How about displacement?

— very roughly speaking:

as wave is linearization, and assumes/predicts certain phase relation,

→ linear wave theory valid for

$|k\tilde{\xi}| < 1$

↳ wave slope

If $k\tilde{\xi} \sim 1$ $\Rightarrow$ expect strongly nonlinear behavior, breaking, mixing etc.

n.b. though for Alfven waves, need add parallel compressibility....

Now $\mathcal{E}(z) = 2\frac{\rho}{2}\overline{\tilde{\xi}^2}$

$= \rho(z)\,\omega^2\tilde{\xi}^2$

Now – ω = const

so – $\tilde{\xi}^2 = \frac{1}{\rho(z)\,\omega^2}\left(\frac{\rho(z)}{\rho_\infty}\right)^{1/2}\mathcal{E}_\infty$

$= \frac{1}{\sqrt{\rho(z)\rho_\infty}}\;\frac{\mathcal{E}_\infty}{\omega^2}$

$\therefore$ displacement drops $\sim \rho(z)^{-1/4}$

as wave propagates into high density region.

but – slope $s \sim |k\tilde{\xi}|$

$k = \frac{\omega}{V_A} = \frac{\omega}{V_{A\infty}}\sqrt{\frac{\rho(z)}{\rho_\infty}}$

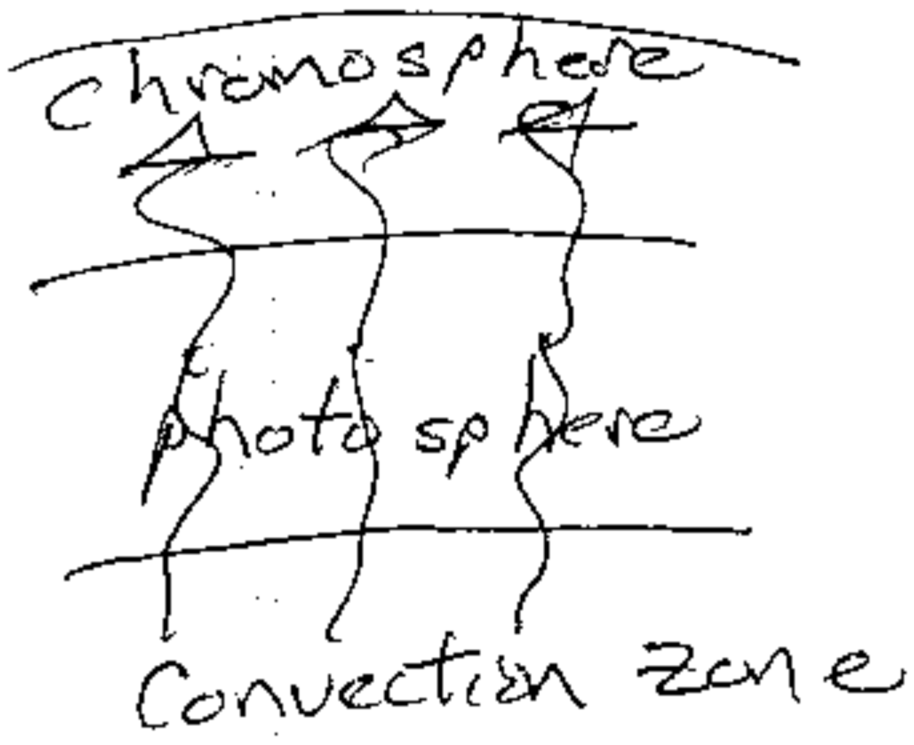
so $\quad |k\tilde{\mathcal{E}}| \sim \dfrac{\omega}{V_{A0}} \sqrt{\dfrac{\rho(z)}{\rho_{00}}} \left(\dfrac{\mathcal{E}_{00}}{\omega}\right)^{1/2} \dfrac{1}{(\rho(z)\rho_{00})^{1/4}}$

$$\sim \rho(z)^{1/4}$$

$\Rightarrow$ wave slope increases in high density region, as V_A changes

$\Rightarrow$ Nonlinearity increases

(2) Sound propagating in chromosphere



$\rho \sim e^{-z/H} \rightarrow$ density decreases with height

Sound waves emitted from convection zone (compressible convection) $\rightarrow$ propagate into chromosphere

Take $T = \text{const} \Rightarrow C_S = \text{const.}$

Then $\quad C_S \mathcal{E} = \text{const.}$

$$\mathcal{E}(z) = \text{const.}$$

and $\quad k = \omega / C_S = \text{const.}$

so $\cancel{\tfrac{1}{2}}\,\underset{\cancel{}}{\rho}\,\dot{\tilde{\xi}}^2 = \text{const}$

$$\rho(z)\,\omega^2\,\tilde{\xi}^2 = \text{const}$$

$$\Rightarrow \quad \tilde{\xi} = \left(\xi_{00}/\rho(z)\,\omega^2\right)^{1/2} \sim 1/(\rho(z))^{1/2}$$

as $\quad k = \text{const}, \qquad k\tilde{\xi} \sim 1/(\rho(z))^{1/2}$

$\Rightarrow$ – wave displacement increases in chromosphere

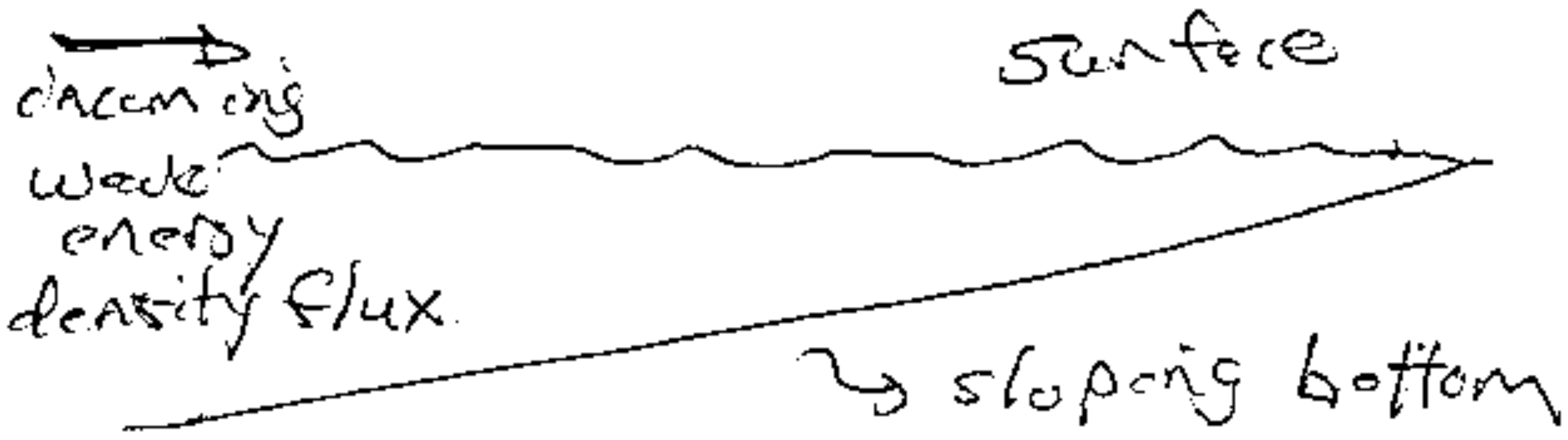
– sound wave simple $\Rightarrow$ wave steepens and can shock

– physical picture is that of a whip $\Rightarrow$ inertia at tip low, due tapering

– constitutes simple argument for chromospheric and possibly coronal heating by sound waves propagating from convection zone into upper layers.

③ The beach....

Consider:



$H = H(x)$

Now, in shallow water
$(\lambda > H)$

$$\frac{\partial h}{\partial t} + \frac{\partial}{\partial x}(vh) = 0$$

$$\frac{\partial v}{\partial t} + v\frac{\partial v}{\partial x} = -g\frac{\partial h}{\partial x}$$

$$v = \cancel{v_0} + \tilde{v} \quad , \qquad h = H + \tilde{h}$$

$$\Rightarrow \quad -i\omega\tilde{h} + ikH\tilde{v} = 0$$
$$-i\omega\tilde{v} = -ikg\tilde{h}$$

$\therefore \Rightarrow \omega^2 = k^2 g H$ is dispersion relation

$\Rightarrow$ analogy with acoustics is obvious

$h \Leftrightarrow \rho \qquad c_s^2 = gH$

$v \Leftrightarrow v \qquad$ etc.

$$\frac{\partial \tilde{v}}{\partial t} = -g \frac{\partial \tilde{h}}{\partial x} \qquad (1)$$

$$\frac{\partial \tilde{h}}{\partial t} = -H \frac{\partial \tilde{v}}{\partial x} \qquad (2)$$

$$\Rightarrow (1) \times \tilde{v} + (2) \times \left(g \frac{\tilde{h}}{H}\right)$$

$$\therefore \quad \frac{\partial}{\partial t} \tilde{v}^2 = -g\, \tilde{v} \frac{\partial \tilde{h}}{\partial x}$$

$$\frac{g}{H} \frac{\partial}{\partial t} \tilde{h}^2 = -g \frac{H}{\cancel{H}} \tilde{h} \frac{\partial \tilde{v}}{\partial x}$$

$$\therefore \quad \frac{\partial}{\partial t}\left(\frac{\tilde{v}^2}{2} + \frac{g\tilde{h}^2}{2H}\right) + \frac{\partial}{\partial x}\left(g \tilde{h} \tilde{v}\right) = 0$$

is energy theorom

$$\Rightarrow \quad \mathcal{E} = \frac{\tilde{v}^2}{2} + \frac{g\tilde{h}^2}{2H}$$ is wave energy density

$$\omega/k = (gH)^{1/2}$$ is wave phase velocity

so .... as no explicit time dependence:

$$\frac{\partial \mathcal{E}}{\partial t} + \nabla \cdot \left(V_{gr}\, \mathcal{E}\right) = 0$$

$$\Rightarrow \quad V_g(x)\,\mathcal{E}(x) = V_\infty \mathcal{E}_\infty = I$$

$$\therefore \quad \sqrt{gH(x)}\;\mathcal{E}(x) = I$$

$\Rightarrow$ as $x \to$ shore, $V_{gr} \downarrow$ so $\mathcal{E}(x)$ must increase

$$\mathcal{E}(x) = \frac{\tilde{v}^2}{2} + \frac{g\tilde{h}^2}{2H} = \overline{\tilde{v}^2}$$

$\to$ horizontal displacement

$$\tilde{v} = \frac{\partial \xi}{\partial t}$$

$$\mathcal{E}(x) = \rho_0 \omega^2 \overline{\tilde{\xi}^2} \qquad \text{and } \rho_0\omega^2 = \text{const, here}$$

$$\Rightarrow \quad \tilde{\xi}_{rms} \sim \left(I \big/ \rho_0 \omega^2 \sqrt{gH(x)} \right)^{1/2} \sim H(x)^{-1/4}$$

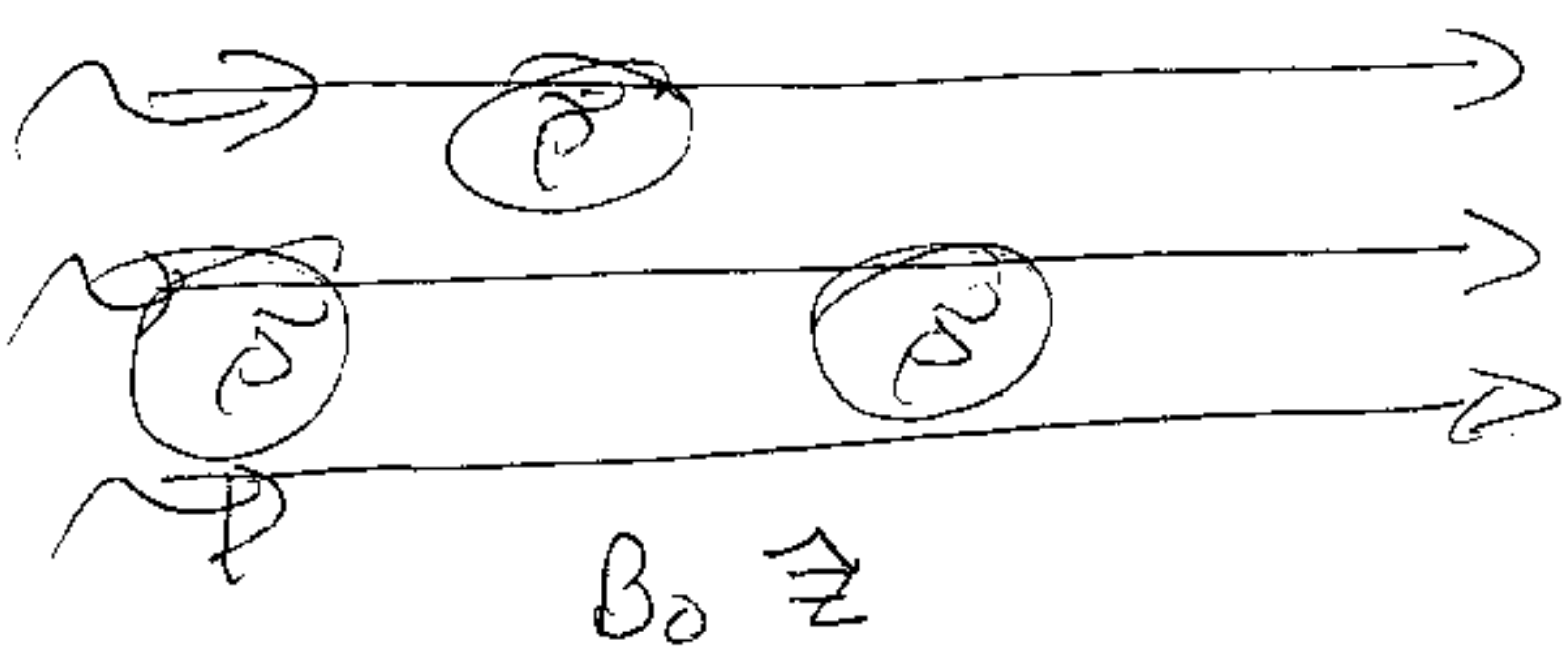
for profile $H(x)$, can deduce displacement profile

N.B. If $H = H(x,y)$, wavefronts align with bottom depth, via refraction.

i.e. $$\frac{d\underline{k}}{dt} = -\frac{\partial}{\partial \underline{x}}\,\omega = -\underline{\nabla}\left[(gH(x,y))^{1/2} k \right]$$

4) Alfven Waves in Random Medium

Consider straight $\underline{B_0}$ threading medium with space-time dependent inhomogeneities ρ_0



i.e. $\rho_0 + \tilde{\rho}$

with $\langle \tilde{\rho}^2 \rangle_{q,\Omega}$ given.

How does spectrum of Alfven waves evolve?

Assume: $|q| << |k|$, $\Omega << |k V_A|$ → clear scale separation between scatterer and scatter-ee

and weak inhomogeneities ($\tilde{\rho} << \rho_0$)

What happens?

$$\frac{d\underline{x}}{dt} = V_{gr} = V_A$$

$$\frac{d\underline{k}}{dt} = -\frac{\partial}{\partial \underline{x}}\cdot\left(k_\parallel V_A\right)$$

$$V_A = \frac{B_0}{\sqrt{4\pi\rho}} \cong V_{A0}\left(1 - \frac{\tilde{\rho}}{2\rho_0}\right)$$

take 1.D for simplicity:

$$\frac{dz}{dt} = V_{A0}\left(1 - \frac{1}{2}\frac{\tilde{\rho}}{\rho_0}\right) = V_{A0}\left(1 - \delta\rho\right)$$

$$\frac{dk_z}{dt} = -\frac{\partial}{\partial z}\left(k_z V_{A0}\left(1 - \frac{\tilde{\rho}}{2\rho_0}\right)\right)$$

refraction

ch

action

$$= \frac{\partial}{\partial z}\left(k_z V_{A0}\frac{\tilde{\rho}}{2\rho_0}\right) = \frac{\partial}{\partial z} k_z V_{A0}\,\delta\rho$$

How does Alfven spectrum respond to this?

$\Rightarrow$ wavekinetics!

$$\frac{\partial N}{\partial t} + \underline{v}_{gr}\cdot\underline{\nabla} N - \frac{\partial}{\partial \underline{x}}\omega\cdot\frac{\partial N}{\partial \underline{k}} = 0$$

$$N = \mathcal{E}(\underline{k},\underline{x})/\omega$$

$$\frac{\partial N}{\partial t} + V_{A0}(1-\delta\rho)\hat{z}\cdot\nabla N + \frac{\partial}{\partial z}\left(k_z V_{A0}\,\delta\rho\right)\frac{\partial N}{\partial k_z} = 0$$

Now $\delta\rho$ is – random variable
– spectrum specified

$\therefore$ for trend, need average

$\Rightarrow$

$$\frac{\partial \langle N \rangle}{\partial t} + \left\langle V_{A0}(1-\delta\rho)\hat{z}\cdot\nabla N \right\rangle + \left\langle \frac{\partial}{\partial z}\left(k_z V_{A0}\,\delta\rho\right)\frac{\partial N}{\partial k_z} \right\rangle = 0$$

and average contributions will come from

$\langle \delta\rho\,\delta N \rangle$ type correlations.

$\therefore$ proceed in spirit of quasi-linear theory.

Using $\underline{\nabla}_r \cdot \underline{V}_r = 0 \Rightarrow$

$$\frac{\partial \langle N \rangle}{\partial t} - \frac{\partial}{\partial z}\left\langle V_{A}\,\delta\rho\,\delta N \right\rangle + \frac{\partial}{\partial k_z}\left\langle \frac{\partial (k_z V_{A0}\,\delta\rho)}{\partial z}\,\delta N \right\rangle = 0$$

where have taken $\langle N \rangle$ indep. of z (uniform 'beam').

Now, to calculate correlations $\langle \delta\rho\,\delta N \rangle$, $\left\langle \frac{\partial}{\partial z}\delta\rho\,\delta N \right\rangle$, use linear response for δN

Linearizing WKE:

homogeneous background

$$\frac{\partial}{\partial t}\delta N + V_A \frac{\partial}{\partial z}\delta N = V_A\,\delta\rho \cancel{\frac{\partial}{\partial z}\langle N\rangle} - \frac{\partial}{\partial z}(k_z V_A \delta\rho)\frac{\partial}{\partial k_z}\langle N\rangle$$

$\Rightarrow$

$$-i(\Omega - q V_A)\,\delta N_{\Omega,q} = -i q k_z V_{A0}\,\delta\rho_{\Omega,q}\,\frac{\partial\langle N\rangle}{\partial k}$$

$$\therefore \quad \delta N_{\Omega,q} = \frac{q k_z V_A\,\delta\rho_{\Omega,q}}{(\Omega - q V_A)}\,\frac{\partial\langle N\rangle}{\partial k}$$

$\Rightarrow$

$$\frac{\partial\langle N\rangle}{\partial t} = \frac{\partial}{\partial k_z} D_{k_z} \frac{\partial\langle N\rangle}{\partial k_z}$$

$\leadsto$ quasi-linear diffusion equation for $\langle N\rangle$

$$D_{k_z} = \sum_{\Omega,q} q^2 k_z^2 V_A^2\, |\delta\rho_{q,\Omega}|^2\, \pi\delta(\Omega - q V_A)$$

$$= \sum_q \pi k_z^2 q^2 V_A^2\, |\delta\rho_{q,\,qV_A}|^2$$

really resonance between

$\frac{\Omega}{q}$ and $V_{gr} = V_A$

$\frac{\Omega}{q}$: strain field phase velocity

V_{gr}: $\hookrightarrow$ packet group velocity.

Note:

- basic gist of answer to question is that random inhomogeneities diffuse $\langle N \rangle$ spectrum in k_z

- physics clear from treating eikonal equation as Langevin equation

i.e.
$$\frac{dk_z}{dt} = -\frac{\partial}{\partial z} V_A(z)$$

$$= -\frac{\partial}{\partial z}\left(V_0\left(1-\frac{1}{2}\frac{\tilde{\rho}}{\rho_0}\right)k_z\right)$$

kick in k_z due inhomog.
$$\frac{dk_z}{dt} = V_{A0} k_z \frac{\partial}{\partial z}\delta\tilde{\rho}$$

stochastic refraction

$$\Rightarrow \langle \delta k_z^2 \rangle \cong D t$$

$$D \cong V_{A0}^2 k_z^2 \left\langle \left|\frac{\partial}{\partial z}\tilde{\rho}\right|^2 \tau_c \right\rangle \equiv D_{k_z}$$

- what is τ_c?

$\Rightarrow$ set by spectrum of inhomogeneities i.e. here Ω, q independent

$\rightarrow$ scatterers not waves $\rightarrow$ width Δq

so if $|\tilde{\rho}|^2_{q,\Omega} = |\tilde{\rho}(q)|^2 \dfrac{\Delta\Omega}{\Omega^2 + (\Delta\Omega)^2}$

then $\tau_c = \min\{1/\Delta\Omega,\ 1/\Delta q V_A\}$

contrast to usual case:

$$\frac{\partial \langle f \rangle}{\partial t} = \frac{\partial}{\partial v} D \frac{\partial \langle f \rangle}{\partial v}$$

$$D = \sum_k \frac{q^2}{m^2} |E_k|^2 \pi \delta(\omega_k - kv)$$

and $\tau_{ac} \sim |k(v_{ph} - v_{gr})|^{-1}$

$\hookrightarrow$ dispersion time for eigenmode packet

- when is QLT applicable? — $\left\{ \begin{array}{l} \text{equivalent to asking} \\ \text{when valid to treat} \\ \text{problem as stochastic} \end{array} \right.$

— basically, ① weak scattering
② resonance overlap

most clearly seen in context of particle

need :

$\tau_{ac} < \tau_{bounce}$

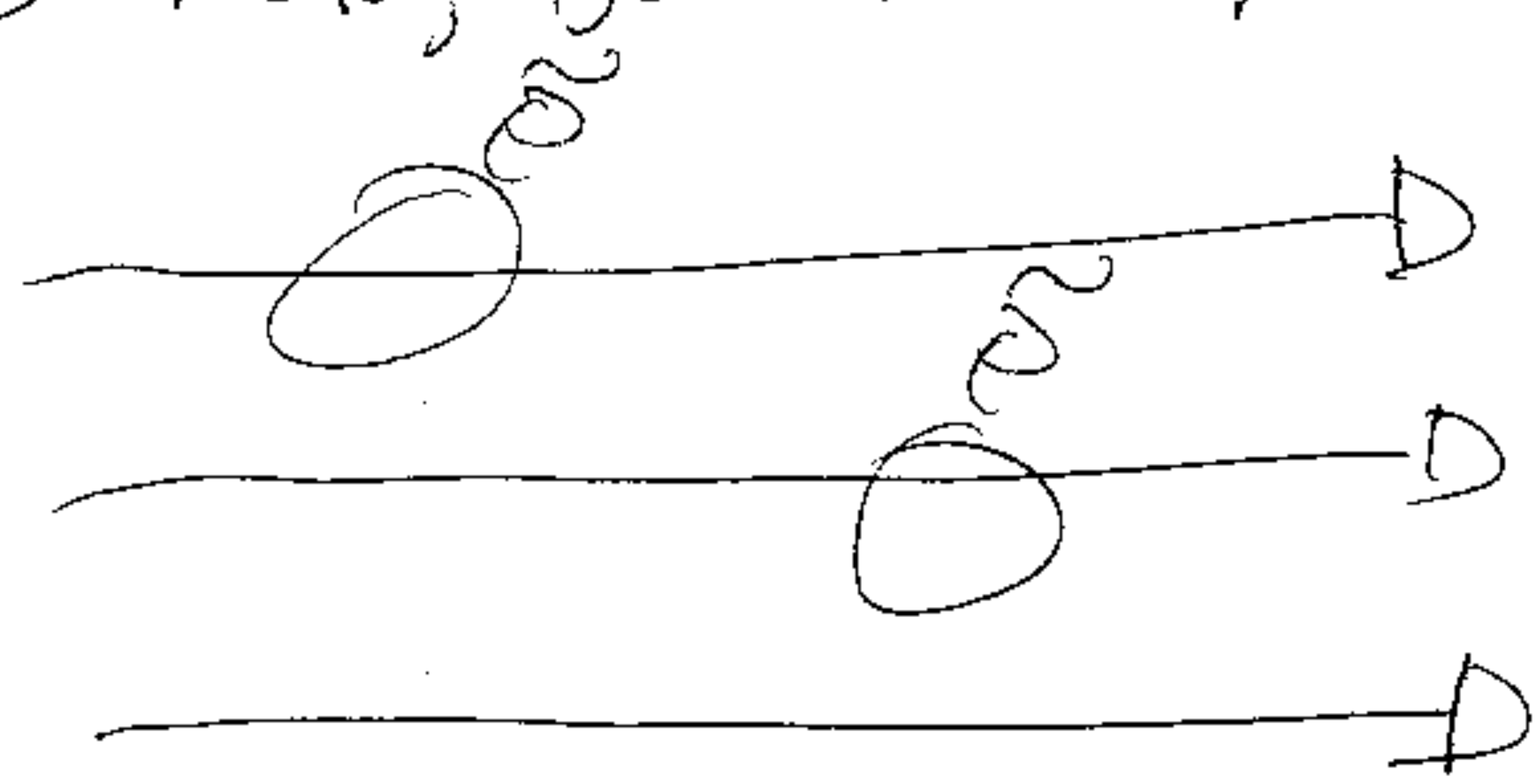
$1/\tau_{bounce} \sim k(q\phi/m)^{1/2}$

so linearization valid.

→ What is the bottom line ? ⇒ spectrum spreads diffusively

in particular, high k_z's generated.

⑤ Now, go one step further ----



$B_0 \hat{z}$

i.e. Alfven waves
ion-acoustic waves

⇒ associate scattering field
— not with randomly prescribed inhomogeneities

– rather, with a field of ion acoustic waves

so in 1D

→ high frequency, short wavelength Alfven waves

and

$$\omega = k_z V_A$$

→ low frequency, longer wavelength ion acoustic waves

$$\Omega^2 = \frac{q^2 C_s^2}{1 + \underbrace{q^2 \lambda_D^2}}$$

↳ dispersion due Debye screening

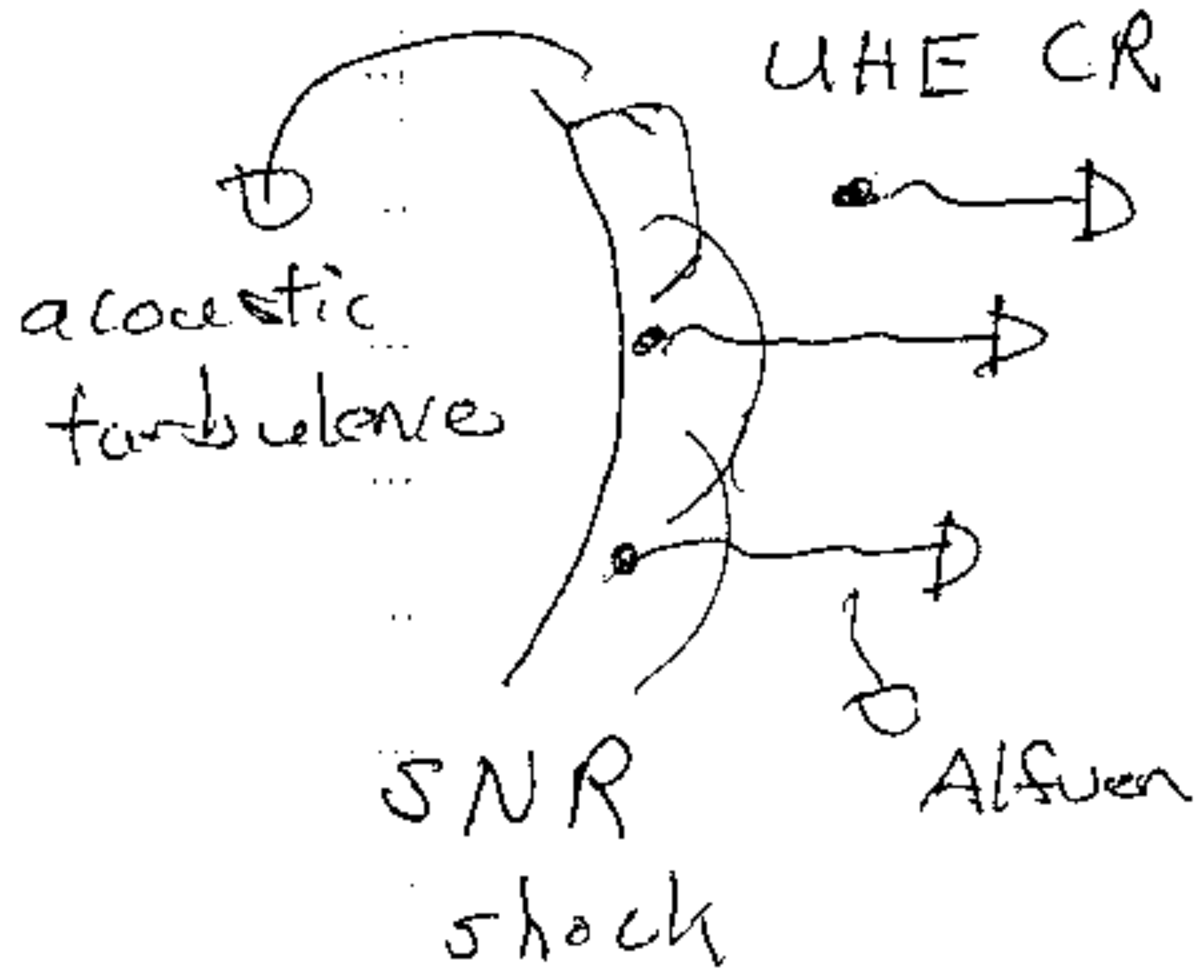
N.B. – this is really a 'nonlinear' problem (very similar to SRS, SBS, Langmuir turbulence)

– but, using eikonal methods, can be treated with linear, quasi-linear methodology

– the "hidden smallness parameter" is scale ratio

$$\frac{\Omega}{\omega} < 1 \;, \quad \frac{q}{k_z} < 1 \,.$$

- what might this be useful for, apart from trial-by-ordeal?



Cerenkov emission
$\Rightarrow$ Alfven waves via
$\Omega_i = k_{\parallel} v_{\parallel i}$

so - have environment where spectrum of Alfven waves co-exists with spectrum of acoustic-type density perturbations.

- interaction could be relevant to process of CR acceleration

N.B. Of course, it is a bit more complicated ...

$\rightarrow$ What new feature enters here?

- eikonal games

- dynamical coupling of high and low frequency waves!

⇒ effective 'pressure' of Alfven waves on acoustic wave !

(f)

⇒ refraction of Alfven waves by acoustic waves, as before

Now, in 1D, recall ion-acoustic wave has:

$$\nabla^2 \tilde{\phi} = 4\pi n_0 |e| \left(\tilde{n}_i - \tilde{n}_e \right)$$

$$\frac{\tilde{n}_e}{n_0} = \frac{|e| \tilde{\phi}}{T_e} \qquad (\omega \ll k v_{Te})$$

⇝ Boltzmann response

and for ions:

$$\frac{\partial n}{\partial t} + \frac{\partial}{\partial x}(n v) = 0$$

$$\frac{\partial v}{\partial t} + v \frac{\partial v}{\partial x} = \left(\frac{|e| E}{m_i} \right) - \frac{1}{nm} \frac{\partial p}{\partial x}$$

$$E = -\partial \phi / \partial x$$

To make easier, treat as 1 fluid, with dispersion later added "by hand"!

$\Rightarrow$

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x}(\rho v) = 0$$

$$\frac{\partial v}{\partial t} + v\frac{\partial v}{\partial x} = -\frac{1}{\rho}\frac{\partial P}{\partial x}$$

i.e. magnetic field irrelevant to parallel-to-$\underline{B}_0$ acoustic wave.

now here, have: $P = P_{Th}$.

With Alfven waves (and 1D), let

$$P \to P_{Th} + P_{AW_{eff}}$$

n.b. for technical reasons, need weak dispersion in Alfven waves

i.e. $\omega^2 = k_{\parallel}^2 V_A^2 / 1 + k^2 c^2/\omega_{pi}^2$

& ignore till needed

but $P_{AW_{eff}} = \mathcal{E}_{AW}$

↳ energy density of Alfven waves

$$= \int dk\, \omega_k N_k$$

↳ Action density of Alfven waves.

so, in linear theory for acoustic wave:

$$\frac{\partial \tilde{\rho}}{\partial t} = -\rho_0 \frac{\partial \tilde{v}}{\partial x}$$

$$\frac{\partial \tilde{v}}{\partial t} = -\frac{1}{\rho_0} \frac{\partial}{\partial x} \left(\tilde{P} + \tilde{P}_{AW} \right)$$

$$\tilde{P} = \gamma P_0 (\tilde{\rho}/\rho_0) \qquad , \qquad \tilde{P}_{AW} = \int dk \, \omega_k \tilde{N}_k$$

from wave kinetic equation

$\Rightarrow$

$$\rho_0 \frac{\partial}{\partial t}\left(\frac{\partial \tilde{v}}{\partial x}\right) = -\frac{\partial^2}{\partial x^2}\left(\gamma P_0 \frac{\tilde{\rho}}{\rho_0} + \tilde{P}_{AW} \right)$$

$$\Rightarrow \quad \frac{\partial^2}{\partial t^2} \tilde{\rho} = \frac{\partial^2}{\partial x^2} \left(\underbrace{\frac{\gamma P_0}{\rho_0}}_{C_s^2} \tilde{\rho} + \tilde{P}_{AW} \right)$$

Now, need calculate $\tilde{P}_{AW}$!

$$\frac{\partial N}{\partial t} + \underline{v}_{gr} \cdot \nabla N - \frac{\partial \omega}{\partial \underline{x}} \cdot \frac{\partial N}{\partial \underline{k}} = 0$$

and linearizing as before $\Rightarrow$

$$\frac{\partial}{\partial t} \delta N + v_A \frac{\partial}{\partial z} \delta N = -\frac{\partial}{\partial z}\left(k_z \; V_A \frac{\tilde{\rho}}{2\rho_0} \right) \frac{\partial \langle N \rangle}{\partial k_z}$$

$\Rightarrow$

$$\delta N_{\Omega, q} = \frac{q k_z V_A}{(\Omega - q V_A)} \left(\frac{\tilde{\rho}}{2\rho_0} \right)_{\Omega, q} \frac{\partial \langle N \rangle}{\partial k}$$

$\therefore$

$$-\Omega^2 \tilde{\rho}_{q \atop \Omega} = -q^2 \left(C_s^2 \tilde{\rho}_{q \atop \Omega} + \int dk_z (k_z V_A) \, \delta N_{\Omega \atop q} \right)$$

$$\left(\Omega^2 - q^2 C_s^2\right) \tilde{\rho}_{q \atop \Omega} = -q^2 \int dk_z \left(k_z V_A\right) \left(\frac{q k_z V_A / 2}{\Omega - q V_A} \right) \frac{\tilde{\rho}_{q, \Omega}}{\rho_0} \frac{\partial \langle N \rangle}{\partial k}$$

$\therefore$

and convenient to write as

$$(\Omega^2 - q^2 c_s^2)\, \tilde{\rho}_{q,\Omega} = -q^2 \int dk_z \underbrace{\left[\frac{k_z V_A \langle N \rangle}{\rho_0}\right]}_{\mathcal{E}/\rho_0 \sim P_{eff}/\rho_0} \left(\frac{q k_z (V_A/2)}{\Omega - q V_A}\right) \frac{1}{\langle N \rangle} \frac{\partial \langle N \rangle}{\partial k} \tilde{\rho}_{q,\Omega}$$

⇒ have recovered a variant of Landau problem:

$$\boxed{(\Omega^2 - q^2 c_s^2) = -q^2 \int dk_z \left(\frac{P_{eff}}{\rho_0}\right)\left(\frac{q k_z (V_A/2)}{\Omega - q V_A}\right) \frac{1}{\langle N \rangle} \frac{\partial \langle N \rangle}{\partial k}}$$

→ effective "radiation pressure" of Alfven waves modifies acoustic mode

→ $V_A = \Omega(q)/q$ resonance

⇒ Landau-like growth / damping

key is $\left.\frac{\partial \langle N \rangle}{\partial k}\right|_{res.}$ ⟺ akin $\left.\frac{\partial f}{\partial v}\right|_{res}$.

Now, can proceed via P.T. if $P_{eff}/P_{th} < 1 \Rightarrow$

$$(\Omega_0 + i\gamma) - q^2 c_s^2 = -q^2 \int dk_z \left(\frac{P_{eff}}{\rho_0}\right) \frac{q k_z (V_A/2)}{\Omega - q V_A} \frac{1}{\langle N \rangle} \frac{\partial \langle N \rangle}{\partial k}$$

$$i 2 q c_s \gamma = q^2 \int dk_z \left(\frac{P_{eff}}{\rho_0}\right) q \frac{k_z V_A}{2} \pi \delta(\Omega - q V_A) \frac{1}{\langle N \rangle} \frac{\partial \langle N \rangle}{\partial k}$$

$$\Rightarrow \gamma_q = \frac{q^2}{c_s} \left(\frac{P_{eff}}{\rho_0}\right) \frac{V_A}{4} \int dk_z \, k_z \, \pi \delta(\Omega - q V_A) \frac{1}{\langle N \rangle} \frac{\partial \langle N \rangle}{\partial k}$$

? ? ?

- point here is that no way to resolve/understand singularity, as Alfven waves are non-dispersive!

- one solution: go outside MHD to introduce dispersion!

i.e. retaining Hall term $\Rightarrow$
(i.e. earlier comment)

$$\omega^2 = k_z^2 V_A^2 / 1 + k_z^2 d_s^2 \qquad d_s^2 = c^2/\omega_{pi}^2$$

∴ then have:

$$\gamma_q = \frac{q^2}{c_s^2}\left(\frac{P_{eff}}{P_0}\right)\frac{V_A}{4}\int dk_z\, k_z\, \pi\delta(\Omega - q V_{gr}(k))\frac{1}{\langle N\rangle}\frac{\partial\langle N\rangle}{\partial k}$$

and resonant k identified! → proceed a/a' Landau

2 lessons:

→ population inversion, i.e. $\left.\frac{\partial\langle N\rangle}{\partial k}\right|_{\text{resonance}} > 0$, needed for growth. Akin to $\partial f/\partial v > 0$.

→ makes important point that non-dispersive waves all strained at same rate, so no Doppler dispersion

⟺ non-dispersive waves steepen → shocks, etc. in MHD

→ can compute $\langle N\rangle$ evolution a/a' QLT, $\Omega(q)$ dispersion relevant.